IB HL Chapter 4 Equations

Example 1 — Self Tutor

Solve for x:

a $x^3 = -27$ **b** $x^4 + 5 = 15$ **c** $5 - 2x^2 = 11$

a
$$x^3 = -27$$
$$\therefore\ x = \sqrt[3]{-27}$$
$$\therefore\ x = -3$$

b
$$x^4 + 5 = 15$$
$$\therefore\ x^4 = 10$$
$$\therefore\ x = \pm\sqrt[4]{10}$$

c
$$5 - 2x^2 = 11$$
$$\therefore\ -2x^2 = 6$$
$$\therefore\ x^2 = -3$$
which has no real solutions as x^2 cannot be negative.

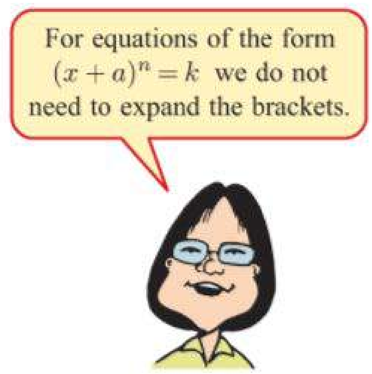
Example 2 — Self Tutor

Solve for x:

a $(x+3)^2 = 36$ **b** $(x-2)^5 = 11$

a
$$(x+3)^2 = 36$$
$$\therefore\ x + 3 = \pm\sqrt{36}$$
$$\therefore\ x + 3 = \pm 6$$
$$\therefore\ x = -3 \pm 6$$
$$\therefore\ x = 3 \text{ or } -9$$

b
$$(x-2)^5 = 11$$
$$\therefore\ x - 2 = \sqrt[5]{11}$$
$$\therefore\ x = 2 + \sqrt[5]{11}$$



Exercise 4.A Solve for x

1) **c** $2x^2 = -10$ **e** $4x^2 - 5 = 15$ **f** $7 - 3x^2 = 19$

2) **h** $x^3 = \frac{8}{27}$ **i** $x^4 = \frac{1}{16}$ **k** $4x^3 + 5 = -19$ **l** $2x^4 - 55 = 107$

3) **g** $(x - \sqrt{2})^2 = 2$ **h** $(2x - \sqrt{3})^2 = 2$ **i** $(2x+1)^2 = 7$

4) **g** $(2x-3)^4 = 15$ **h** $\frac{1}{4}(1-x)^5 = 8$ **i** $14 = 3 - \frac{1}{2}(4-x)^3$

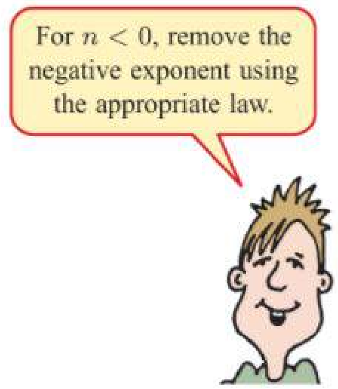
Example 3 Self Tutor

Solve for x:

a $x^{-5} = -\frac{1}{32}$ **b** $x^{-4} = 256$

a
$$x^{-5} = -\frac{1}{32}$$
$$\therefore\ x^5 = -32$$
$$\therefore\ x = \sqrt[5]{-32}$$
$$\therefore\ x = -2$$

b
$$x^{-4} = 256$$
$$\therefore\ x^4 = \frac{1}{256}$$
$$\therefore\ x = \pm\sqrt[4]{\frac{1}{256}}$$
$$\therefore\ x = \pm\frac{1}{4}$$



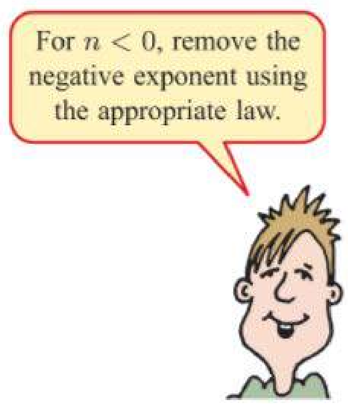
Example 3 Self Tutor

Solve for x:

a $x^{-5} = -\frac{1}{32}$ **b** $x^{-4} = 256$

a
$$x^{-5} = -\frac{1}{32}$$
$$\therefore\ x^5 = -32$$
$$\therefore\ x = \sqrt[5]{-32}$$
$$\therefore\ x = -2$$

b
$$x^{-4} = 256$$
$$\therefore\ x^4 = \frac{1}{256}$$
$$\therefore\ x = \pm\sqrt[4]{\frac{1}{256}}$$
$$\therefore\ x = \pm\frac{1}{4}$$



5) **e** $x^{-4} = \frac{1}{16}$ **f** $x^{-3} = -64$ **g** $(x+1)^{-2} = -4$ **h** $(2x-5)^{-3} = \frac{1}{5}$

Exercise 4B

Example 4 — Self Tutor

Solve for x using the null factor law:

a $3x(x-5)=0$

b $(x-4)(3x+7)=0$

a
$$3x(x-5)=0$$
$$\therefore\ 3x=0 \text{ or } x-5=0$$
$$\therefore\ x=0 \text{ or } 5$$

b
$$(x-4)(3x+7)=0$$
$$\therefore\ x-4=0 \text{ or } 3x+7=0$$
$$\therefore\ x=4 \text{ or } 3x=-7$$
$$\therefore\ x=4 \text{ or } -\tfrac{7}{3}$$

1 Solve using the null factor law:

a $4x=0$ **b** $ab=0$ **c** $2xy=0$ **d** $3abc=0$

2) **i** $4(5-x)^2=0$ **j** $-3(3x-1)^2=0$

2) **k** $x(x+1)(x-2)=0$ **l** $3(x+2)(x+4)(2x-1)=0$

Example 5 — Self Tutor

Solve for x:

a $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

a
$$3x^2 + 5x = 0$$
$$\therefore \; x(3x+5) = 0$$
$$\therefore \; x = 0 \text{ or } 3x + 5 = 0$$
$$\therefore \; x = 0 \text{ or } x = -\tfrac{5}{3}$$

b
$$x^2 = 5x + 6$$
$$\therefore \; x^2 - 5x - 6 = 0$$
$$\therefore \; (x-6)(x+1) = 0$$
$$\therefore \; x = 6 \text{ or } -1$$

Example 6 — Self Tutor

Solve for x:

a $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a
$$4x^2 + 1 = 4x$$
$$\therefore \; 4x^2 - 4x + 1 = 0$$
$$\therefore \; (2x-1)^2 = 0$$
$$\therefore \; x = \tfrac{1}{2}$$

b
$$6x^2 = 11x + 10$$
$$\therefore \; 6x^2 - 11x - 10 = 0$$
$$\therefore \; (2x-5)(3x+2) = 0$$
$$\therefore \; x = \tfrac{5}{2} \text{ or } -\tfrac{2}{3}$$

Caution:

- Do not be tempted to divide both sides by an expression involving x.
  If you do this then you may lose one of the solutions.
  For example, consider $x^2 = 5x$.

Correct solution
$$x^2 = 5x$$
$$\therefore \; x^2 - 5x = 0$$
$$\therefore \; x(x-5) = 0$$
$$\therefore \; x = 0 \text{ or } 5$$

Incorrect solution
$$x^2 = 5x$$
$$\therefore \; \frac{x^{\cancel{2}}}{\cancel{x}} = \frac{5\cancel{x}}{\cancel{x}}$$
$$\therefore \; x = 5$$

We cannot divide by 0. In dividing both sides by x, we assume $x \neq 0$. For this reason, the solution $x = 0$ is lost.

- Be careful when taking square roots of both sides of an equation.
  If you do this then you may lose one of the solutions.
  For example, consider $(2x-7)^2 = (x+1)^2$.

Correct solution

$$\begin{aligned}(2x-7)^2 &= (x+1)^2\\ \therefore\ (2x-7)^2-(x+1)^2 &= 0\\ \therefore\ (2x-7+x+1)(2x-7-x-1) &= 0\\ \therefore\ (3x-6)(x-8) &= 0\\ \therefore\ x &= 2 \text{ or } 8\end{aligned}$$

Incorrect solution

$$\begin{aligned}(2x-7)^2 &= (x+1)^2\\ \therefore\ 2x-7 &= x+1\\ \therefore\ x &= 8\end{aligned}$$

Exercise 4C.1

1) **a** $4x^2+7x=0$

b $2x^2-11x=0$

i $x^2+8x=33$

j $4x=70-2x^2$

Example 7 **Self Tutor**

Solve for x: $3x+\frac{2}{x}=-7$

$$\begin{aligned}3x+\frac{2}{x} &= -7\\ \therefore\ 3x^2+2 &= -7x && \{\text{multiplying both sides by } x\}\\ \therefore\ 3x^2+7x+2 &= 0 && \{\text{making the RHS} = 0\}\\ \therefore\ (x+2)(3x+1) &= 0 && \{\text{factorising}\}\\ \therefore\ x &= -2 \text{ or } -\tfrac{1}{3}\end{aligned}$$

3 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$

b $(x+2)(1-x) = -4$

e $2x - \dfrac{1}{x} = -1$

f $\dfrac{x+3}{1-x} = -\dfrac{9}{x}$

An alternative method is to rewrite the equation in the form $(x-h)^2 = k$. We refer to this process as "completing the square".

Start with the quadratic equation in the form $ax^2 + bx + c = 0$.

Step 1: If $a \neq 1$, divide both sides by a.

Step 2: Rearrange the equation so that only the constant is on the RHS.

Step 3: Add to both sides $\left(\dfrac{\text{coefficient of } x}{2}\right)^2$.

Step 4: Factorise the LHS.

Step 5: Use the rule: If $X^2 = k$ then $X = \pm\sqrt{k}$.

Example 8 **Self Tutor**

Solve exactly for x: $x^2 + 4x + 1 = 0$

$x^2 + 4x + 1 = 0$

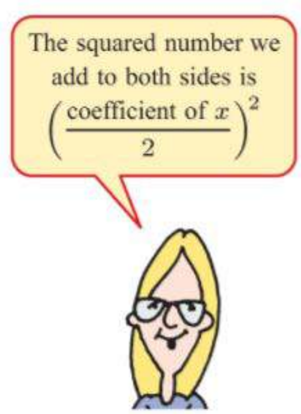
$\therefore\ x^2 + 4x = -1$ {writing the constant on the RHS}

$\therefore\ x^2 + 4x + 2^2 = -1 + 2^2$ {completing the square}

$\therefore\ (x+2)^2 = 3$ {factorising the LHS}

$\therefore\ x + 2 = \pm\sqrt{3}$

$\therefore\ x = -2 \pm \sqrt{3}$



Exercise 4C.2 Solve exactly by completing the square

g $x^2 + 6x = 2$

h $x^2 + 10 = 8x$

i $x^2 + 6x = -11$

2 Solve exactly by completing the square:

a $2x^2 + 4x + 1 = 0$

b $2x^2 - 10x + 3 = 0$

c $3x^2 + 12x + 5 = 0$

d $3x^2 = 6x + 4$

e $5x^2 - 15x + 2 = 0$

f $4x^2 + 4x = 5$

3) Solve for x

a $3x - \frac{2}{x} = 4$

b $1 - \frac{1}{x} = -5x$

c $3 + \frac{1}{x^2} = -\frac{5}{x}$

Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 10 **Self Tutor**

Solve for x:

a $x^2 - 2x - 6 = 0$

b $2x^2 + 3x - 6 = 0$

a $x^2 - 2x - 6 = 0$ has $a = 1$, $b = -2$, $c = -6$

$\therefore\ x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$

$\therefore\ x = \dfrac{2 \pm \sqrt{28}}{2}$

$\therefore\ x = \dfrac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$

b $2x^2 + 3x - 6 = 0$ has $a = 2$, $b = 3$, $c = -6$

$\therefore\ x = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$

$\therefore\ x = \dfrac{-3 \pm \sqrt{57}}{4}$

Exercise 4C.3

1) Use the quadratic formula to solve exactly for x:

g $3x^2 - 5x - 1 = 0$ **h** $-x^2 + 4x + 6 = 0$ **i** $-2x^2 + 7x - 2 = 0$

2) Use the quadratic formula to solve exactly for x:

d $(3x + 1)^2 = -2x$ **e** $(x + 3)(2x + 1) = 9$ **f** $(2x + 3)(2x - 3) = x$

g $\dfrac{x-1}{2-x} = 2x+1$ **h** $x - \dfrac{1}{x} = 1$ **i** $2x - \dfrac{1}{x} = 3$

THE DISCRIMINANT OF A QUADRATIC

We can determine how many real solutions a quadratic equation has, without actually solving the equation.

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
$$x = \frac{-b+\sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b-\sqrt{\Delta}}{2a}.$$
- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution**, which we call a **repeated root**.
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 11 Self Tutor

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$ **b** $3x^2 - 4x - 2 = 0$

a $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(2)(3)$
$= -20$

Since $\Delta < 0$, there are no real roots.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(3)(-2)$
$= 40$

Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

Exercise 4C.4

1 Consider the quadratic equation $x^2 - 7x + 9 = 0$.

- **a** Find the discriminant.
- **b** Hence state the nature of the roots of the equation.
- **c** Check your answer to **b** by solving the equation.

4 Using the discriminant only, state the nature of the solutions of:

- **a** $x^2 + 7x - 3 = 0$
- **b** $x^2 - 3x + 2 = 0$
- **c** $3x^2 + 2x - 1 = 0$

5 Using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

- **a** $6x^2 - 5x - 6 = 0$
- **b** $2x^2 - 7x - 5 = 0$
- **c** $3x^2 + 4x + 1 = 0$

Example 12 **Self Tutor**

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root **b** two distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$$\therefore \Delta = b^2 - 4ac = (-2)^2 - 4(1)(m) = 4 - 4m$$

a For a repeated root

$\Delta = 0$

$\therefore 4 - 4m = 0$

$\therefore 4 = 4m$

$\therefore m = 1$

b For two distinct real roots

$\Delta > 0$

$\therefore 4 - 4m > 0$

$\therefore -4m > -4$

$\therefore m < 1$

c For no real roots

$\Delta < 0$

$\therefore 4 - 4m < 0$

$\therefore -4m < -4$

$\therefore m > 1$

6 For each of the following quadratic equations, find the discriminant Δ in simplest form. Hence find the values of m for which the equation has:

i a repeated root **ii** two distinct real roots **iii** no real roots.

a $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0, \quad m \neq 0$

7 The quadratic equation $4x^2 + kx + (3 - k) = 0$ has a repeated root. Find the possible values of k, and the repeated root in each case.

Example 13 — Self Tutor

Consider the equation $kx^2 + (k+3)x = 1$, $k \neq 0$. Find the discriminant Δ and draw its sign diagram. Hence find the value(s) of k for which the equation has:

a two distinct real roots
b two real roots
c a repeated root
d no real roots.

$kx^2 + (k+3)x - 1 = 0$ has $a = k$, $b = (k+3)$, and $c = -1$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (k+3)^2 - 4(k)(-1) \\ &= k^2 + 6k + 9 + 4k \\ &= k^2 + 10k + 9 \\ &= (k+9)(k+1)\end{aligned}$$

So, Δ has sign diagram:

a For two distinct real roots, $\Delta > 0$ $\therefore$ $k < -9$ or $k > -1$, $k \neq 0$.
b For two real roots, $\Delta \geqslant 0$ $\therefore$ $k \leqslant -9$ or $k \geqslant -1$, $k \neq 0$.
c For a repeated root, $\Delta = 0$ $\therefore$ $k = -9$ or $k = -1$.
d For no real roots, $\Delta < 0$ $\therefore$ $-9 < k < -1$.

8 For each of the following quadratic equations, find the discriminant Δ and hence draw its sign diagram. Hence find the value(s) of k for which the equation has:

i two distinct real roots
ii two real roots
iii a repeated root
iv no real roots.

a $2x^2 + kx - k = 0$

b $kx^2 - 2x + k = 0$, $k \neq 0$

d $2x^2 + (k-2)x + 2 = 0$

THE SUM AND PRODUCT OF THE ROOTS

If $ax^2 + bx + c = 0$ has roots α and β, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Example 14 — Self Tutor

Find the sum and product of the roots of $25x^2 - 20x + 1 = 0$.
Check your answer by solving the quadratic.

If α and β are the roots then $\alpha + \beta = -\frac{b}{a} = \frac{20}{25} = \frac{4}{5}$

and $\alpha\beta = \frac{c}{a} = \frac{1}{25}$

Check: $25x^2 - 20x + 1 = 0$ has roots

$$\frac{20 \pm \sqrt{400 - 4(25)(1)}}{50} = \frac{20 \pm \sqrt{300}}{50} = \frac{20 \pm 10\sqrt{3}}{50} = \frac{2 \pm \sqrt{3}}{5}$$

These have sum $= \frac{2+\sqrt{3}}{5} + \frac{2-\sqrt{3}}{5} = \frac{4}{5}$ ✓

and product $= \left(\frac{2+\sqrt{3}}{5}\right)\left(\frac{2-\sqrt{3}}{5}\right) = \frac{4-3}{25} = \frac{1}{25}$ ✓

Exercise 4C.5

1) For each of the following quadratic equations:

I Find the sum and product of the roots

II Check your answer by solving the quadratic

a $x^2 + 4x - 21 = 0$

b $4x^2 - 12x + 5 = 0$

Example 15 — Self Tutor

The roots of the equation $4x^2 + 5x - 1 = 0$ are α and β.
Find a quadratic equation with roots 3α and 3β.

If α and β are the roots of $4x^2 + 5x - 1 = 0$, then $\alpha + \beta = -\frac{5}{4}$ and $\alpha\beta = -\frac{1}{4}$.

For the quadratic equation with roots 3α and 3β,

the sum of the roots $= 3\alpha + 3\beta = 3(\alpha + \beta) = 3(-\frac{5}{4}) = -\frac{15}{4}$

and the product of the roots $= (3\alpha)(3\beta) = 9\alpha\beta = 9(-\frac{1}{4}) = -\frac{9}{4}$

So, we have $-\dfrac{b}{a} = -\dfrac{15}{4}$ and $\dfrac{c}{a} = -\dfrac{9}{4}$.

The simplest solution is $a = 4$, $b = 15$, $c = -9$

$\therefore$ the quadratic equation is $4x^2 + 15x - 9 = 0$.

All quadratics of the form $k(4x^2 + 15x - 9) = 0$, $k \in \mathbb{R}$, $k \neq 0$, have roots 3α and 3β.

6 Find a quadratic equation with roots:

a 3 and -5

b $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

10 The roots of the equation $x^2 - 6x + 7 = 0$ are α and β.
Find a quadratic equation with roots $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$.

The highest power of x in a polynomial equation is called its **degree**.

If a polynomial equation has degree n then it may have up to n real solutions.

Example 16 — Self Tutor

Use technology to solve:

a $2x^2 + 4x = 7$

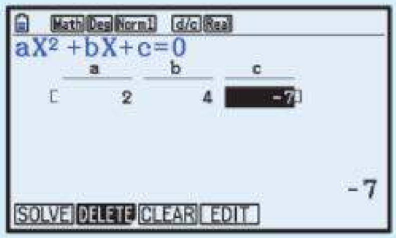
b $x^3 + 4x^2 = 6 - x$

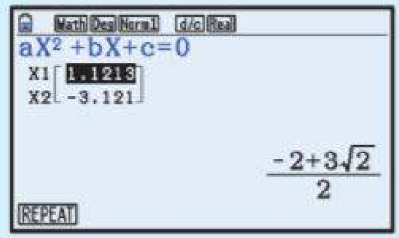
a $2x^2 + 4x = 7$

$\therefore\ 2x^2 + 4x - 7 = 0$

Using technology,

$x \approx 1.12$ or -3.12



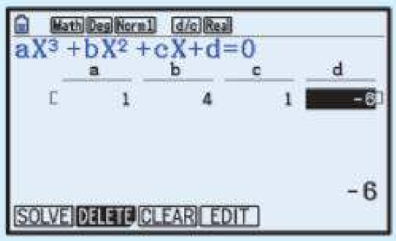


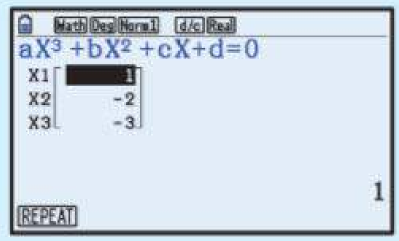
b $x^3 + 4x^2 = 6 - x$

$\therefore\ x^3 + 4x^2 + x - 6 = 0$

Using technology,

$x = -3, -2,$ or 1





Exercise 4D

1) Use technology to solve

a $x^2 + 9x + 14 = 0$

b $x^2 - 8x + 16 = 0$

3) Use technology to solve

c $x^3 - x^2 - 14x + 24 = 0$

d $-x^3 + x^2 - 2x + 2 = 0$

5) Use technology to solve

a $x(x^2 - 1) = 2x$

b $(x - 2)(x + 1) = x^3$